

# Stability analysis of solutions to equilibrium problems and applications in economics

Solutions to  
equilibrium  
problems

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## Abstract

**Purpose** – The purpose of this paper is to study the Hölder calmness of solutions to equilibrium problems and apply it to economics.

**Design/methodology/approach** – The authors obtain the Hölder calmness by using an effective approach. More precisely, under the key assumption of strong convexity, sufficient conditions for the Hölder continuity of solution maps to equilibrium problems are established.

**Findings** – A new result in stability analysis for equilibrium problems and applications in economics is archived.

**Originality/value** – The authors confirm that the paper has not been published previously, is not under consideration for publication elsewhere and is not being simultaneously submitted elsewhere.

**Keywords** Calmness stability, Equilibrium problem, H5C, Mean-variance portfolio, Nash equilibria, Strong convexity

**Paper type** Research paper

## 1. Introduction

Many important problems such as optimization problems, variational inequality problems, complementarity problems, Nash equilibrium problems, minimax problems, fixed-point and coincidence-point problems and traffic network problems are considered as special cases of an equilibrium problem (Blum and Oettli, 1994). This implies possibility of a wide application of results in the equilibrium problem theory to several important fields,

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including economics, physics (especially, mechanics), engineering, transportation, sociology, chemistry, biology, etc. (Kassay and Radulescu, 2018).

Existence conditions of solutions for the equilibrium problem and its generalizations are the first and most developed topic. There are a lot of works devoted to this topic in the literature, see e.g. Ansari *et al.* (2001), Castellani *et al.* (2010), Hai and Khanh (2007), Jafari *et al.* (2017), Hai *et al.* (2009), Sadeqi and Alizadeh (2011) and Alleche and Rădulescu (2016), and the references therein. The second one is stability analysis of solutions. Stability conditions, especially the lower semicontinuity and Hölder continuity of solution maps, for such problems have been extensively studied and received increasing attention from many researchers so far (Bianchi and Rita, 2006; Anh and Khanh, 2008; Anh and Khanh, 2010; Kimura and Yao, 2008; Kimura and Yao, 2008; Li *et al.*, 2013; Anh *et al.*, 2018; and references therein).

The paper aims at investigating the stability analysis in the sense of Hölder calmness of the solution maps to equilibrium problems. To be more precise, sufficient conditions for the solution maps to be Hölder calm are established. At the end of the paper, we present applications of the main results in economics. Namely, the Hölder calmness of solution maps to mean-variance portfolio and Nash equilibrium problems is derived.

The rest of the paper is organized as follows. Section 2 states the equilibrium problem setting and recalls some definitions and their properties needed in what follows. Sufficient conditions for the Hölder calmness of the solution maps are established in Section 3. In Section 4, we discuss the Hölder calmness of solution maps to mean-variance portfolio and Nash equilibrium problems.

## 2. Preliminaries

In this paper, we use  $d(\cdot, \cdot)$  for the metric in metric spaces. For two subsets  $A, B \subseteq X$ , we use the following notations:

$$d(a, B) := \inf_{b \in B} d(a, b),$$

$$H^*(A, B) := \sup_{a \in A} d(a, B),$$

$$H(A, B) := \max\{H^*(A, B), H^*(B, A)\},$$

$$\rho(A, B) := \sup_{a \in A, b \in B} d(a, b).$$

Denote  $\mathbb{R}_+$  is the set of the nonnegative real numbers and  $\mathbb{B}(x, r)$  is the closed ball of radius  $r \geq 0$  and is centered at  $x$ .  $\text{int}A$ ,  $\text{conv}(A)$  and  $\text{diam } A := \sup_{x, z \in A} d(x, z)$  stand for the interior, the

convex hull and the diameter, respectively, of a subset  $A$ . For a set-valued map  $G : X \rightrightarrows Y$ ,  $\text{gph}G := \{(x, y) \in X \times Y : y \in G(x)\}$  is the graph of  $G$ . Recall that  $X$  is called a metric linear space if and only if it is both a metric space and a linear space and the metric  $d$  of  $X$  is translation invariant (i.e.  $d(x + z, y + z) = d(x, y), \forall x, y, z \in X$ ) and, for any convergent sequences  $(\lambda_m)$  in  $\mathbb{R}$  and  $(x^m)$  in  $X$ , we have  $\lim_m (\lambda_m x^m) = \left(\lim_m \lambda_m\right) \left(\lim_m x^m\right)$ .

From now on, unless otherwise stated, let  $X$  be a metric linear space,  $\Lambda, M$  be metric spaces and  $A \subseteq X$  be nonempty. Let  $K: \Lambda \rightrightarrows A$  be nonempty-convex-valued and  $\varphi: A \times A \times M \rightarrow \mathbb{R}$ . For  $(\lambda, \mu) \in \Lambda \times M$ , we consider the following parametric equilibrium problem, i.e. the family of the corresponding problems when  $(\lambda, \mu)$  varies in  $\Lambda \times M$ ,

(EP) find  $\bar{x} \in K(\lambda)$  such that  $\varphi(\bar{x}, y, \mu) \geq 0$  for all  $y \in K(\lambda)$ .

For each  $(\lambda, \mu) \in \Lambda \times M$ , denote the solution set of (EP) by  $S(\lambda, \mu)$ , i.e.:

$$S(\lambda, \mu) := \{x \in K(\lambda) : \varphi(x, y, \mu) \geq 0, \forall y \in K(\lambda)\}.$$

As existence of solutions has been studied much in the literature, we do not include existence investigations and always assume that  $S(\lambda, \mu)$  is nonempty in the neighborhood of the considered point.

We first recall some notions needed in the sequel.

*Definition 2.1.* Let  $n, \gamma > 0$  and  $\theta \geq 0$ . It is said that:

- a function  $g: X \rightarrow \mathbb{R}$  is  $n, \gamma$ -Hölder continuous at  $\bar{x} \in X$  if there is a neighborhood  $U$  of  $\bar{x}$  such that, for all  $x_1, x_2 \in U$ :

$$d(g(x_1), g(x_2)) \leq nd^\gamma(x_1, x_2);$$

- a function  $g: X \rightarrow \mathbb{R}$  is  $n, \gamma$ -Hölder calm at  $\bar{x} \in X$  if there is a neighborhood  $U$  of  $\bar{x}$  such that, for all  $x \in U$ :

$$d(g(x), g(\bar{x})) \leq nd^\gamma(x, \bar{x});$$

- a function  $g: X \times X \times M \rightarrow \mathbb{R}$  is  $n, \gamma$ -Hölder calm at  $\bar{\mu} \in M$ ,  $\theta$ -uniformly over a subset  $B \subseteq X$  if there is a neighborhood  $U$  of  $\bar{\mu}$  such that, for all  $\mu \in U$  and  $x, y \in B: x \neq y$ :

$$d(g(x, y, \mu), g(x, y, \bar{\mu})) \leq nd^\gamma(\mu, \bar{\mu})d^\theta(x, y);$$

if  $\theta = 0$ , we say that  $g(x, y, \cdot)$  is  $n, \gamma$ -Hölder calm at  $\bar{\mu}$ , uniformly over  $B$ ; and

- a multifunction  $K: \Lambda \rightrightarrows X$  is  $n, \gamma$ -Hölder calm at  $\bar{\lambda} \in \Lambda$  if there exists a neighborhood  $N$  of  $\bar{\lambda}$  such that, for all  $\lambda \in N$ :

$$H(K(\lambda), K(\bar{\lambda})) \leq d^\gamma(\lambda, \bar{\lambda}).$$

If  $\gamma = 1$ , then Hölder continuity is called Lipschitz continuity.

We say that a certain property is satisfied in a subset  $B \subseteq X$  if and only if it is satisfied at every point of  $B$ .

*Definition 2.2.* Let  $g: X \rightarrow \mathbb{R}$ ,  $B \subseteq X$ , and  $h, \beta$  be positive.

- $g$  is called  $h, \beta$ -strongly convex on a convex subset  $B$  if and only if, for all  $x_1, x_2 \in B$  and  $t \in (0, 1)$ :

$$g((1-t)x_1 + tx_2) \leq (1-t)g(x_1) + tg(x_2) - ht(1-t)d^\beta(x_1, x_2).$$

- $g$  is called  $h, \beta$ -strongly convex-like in  $B$  ( $B$  not necessarily convex) if and only if, for all  $x_1, x_2 \in B$  and  $t \in (0, 1)$ , there is  $z \in B$  such that:

$$g(z) \leq (1 - t)g(x_1) + tg(x_2) - ht(1 - t)d^\beta(x_1, x_2).$$

Recall that a function  $g : X \times X \rightarrow \mathbb{R}$  is called monotone on  $B \subseteq X$  if:

$$g(x, y) + g(y, x) \leq 0, \forall x, y \in B.$$

In what follows, we use the following assumptions that play an important role in investigating conditions for the Hölder calmness of solution maps to the equilibrium problems:

- H1.  $K$  is  $l, \alpha$ -Hölder calm on  $\Lambda$ .
- H2. The map  $y \mapsto \varphi(x, y, \mu)$  is  $h, \beta$ -strongly convex as well as  $m, 1$ -Hölder continuous in  $\text{conv}(K(\Lambda))$ .
- H3. The map  $(x, y) \mapsto \varphi(x, y, \mu)$  is monotone on  $K(\Lambda) \times K(\Lambda)$  for all  $\mu \in M$ .
- H4. The map  $\mu \mapsto \varphi(x, y, \mu)$  is  $n, \gamma$ -Hölder calm on  $M$ ,  $\theta$ -uniformly over  $K(\Lambda)$ .

### 3. Hölder calmness of solutions

In this section, we state the main results of the paper. Namely, we establish sufficient conditions for the Hölder calmness of the solution maps to equilibrium problems.

*Theorem 3.1.* Assume that all assumptions (H1–H4) are satisfied with  $\theta < \beta$ . Then, the solution map  $S$  is single-valued and Hölder calm on  $\Lambda \times M$ .

*Proof.*

Let  $(\bar{\lambda}, \bar{\mu}) \in \Lambda \times M$  be arbitrary, we need to prove that  $S$  is Hölder calm at  $(\bar{\lambda}, \bar{\mu})$ . We divide the proof into three steps.

*Step 1.* For all  $x_{11} \in S(\bar{\lambda}, \bar{\mu})$  and  $x_{21} \in S(\bar{\lambda}, \bar{\mu})$ , we claim that:

$$d_1 := d(x_{11}, x_{21}) \leq \left(\frac{4ml}{h}\right)^{\frac{1}{\beta}} d^\beta(\bar{\lambda}, \lambda). \tag{1}$$

By the definition of the solution set, we have, for all  $y \in K(\bar{\lambda})$  and  $z \in K(\lambda)$ :

$$\min\{\varphi(x_{11}, y, \bar{\mu}), \varphi(x_{21}, z, \bar{\mu})\} \geq 0. \tag{2}$$

The Hölder calmness of  $K$  leads to the existence of  $x_1 \in K(\bar{\lambda})$  and  $x_2 \in K(\lambda)$  such that:

$$\max\{d(x_{11}, x_2), d(x_{21}, x_1)\} \leq ld^\alpha(\bar{\lambda}, \lambda). \tag{3}$$

Letting  $\hat{x} = \frac{1}{2}(x_{11} + x_{21})$ , it follows from the strong convexity in H2 that:

$$\varphi(x_{11}, \hat{x}, \bar{\mu}) \leq \frac{1}{2} \varphi(x_{11}, x_{11}, \bar{\mu}) + \frac{1}{2} \varphi(x_{11}, x_{21}, \bar{\mu}) - \frac{1}{4}hd_1^\beta. \tag{4}$$

By virtue of the monotonicity of  $\varphi$  and  $x_{11} \in S(\bar{\lambda}, \bar{\mu})$ , one gets:

$$f(x_{11}, x_{11}, \bar{\mu}) = 0$$

$$f(x_{11}, x_{21}, \bar{\mu}) \leq -f(x_{21}, x_{11}, \bar{\mu}).$$

Hence, [equation \(4\)](#) implies that:

$$\frac{1}{4}hd_1^\beta \leq -\frac{1}{2}\varphi(x_{21}, x_{11}, \bar{\mu}) - \varphi\left(x_{11}, \frac{x_{11} + x_{21}}{2}, \bar{\mu}\right). \quad (5)$$

Letting  $z = x_2$  and  $y = \frac{1}{2}(x_{11} + x_1)$  in [equation \(2\)](#), one has:

$$\min\left\{\frac{1}{2}\varphi(x_{21}, x_2, \bar{\mu}), \varphi\left(x_{11}, \frac{x_{11} + x_1}{2}, \bar{\mu}\right)\right\} \geq 0 \quad (6)$$

From [equations \(5\)](#) and [\(6\)](#), we arrive at:

$$\begin{aligned} \frac{1}{4}hd_1^\beta &\leq \frac{1}{2}[\varphi(x_{21}, x_2, \bar{\mu}) - \varphi(x_{21}, x_{11}, \bar{\mu})] \\ &\quad + \left[\varphi\left(x_{11}, \frac{x_{11} + x_1}{2}, \bar{\mu}\right) - \varphi\left(x_{11}, \frac{x_{11} + x_{21}}{2}, \bar{\mu}\right)\right]. \end{aligned}$$

Hence, using the  $m$ .1-Hölder calmness in  $H2$  and [equation \(3\)](#), we obtain:

$$\begin{aligned} \frac{1}{4}hd_1^\beta &\leq \frac{1}{2}md(x_{11}, x_2) + \frac{1}{2}md(x_{21}, x_1) \\ &\leq \frac{1}{2}mld^\alpha(\bar{\lambda}, \lambda) + \frac{1}{2}mld^\alpha(\bar{\lambda}, \lambda), \end{aligned}$$

that is:

$$\frac{1}{4}hd_1^\beta \leq mld^\alpha(\bar{\lambda}, \lambda).$$

Hence, we obtain (1).

*Step 2.* We will show that, for all  $x_{21} \in S(\lambda, \bar{\mu})$  and  $x_{22} \in S(\lambda, \mu)$ :

$$d_2 := d(x_{21}, x_{22}) \leq \left(\frac{n}{h}\right)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma}{\beta-\theta}}(\bar{\mu}, \mu). \quad (7)$$

By the definition of the solution, one has, for all  $y, z \in K(\lambda)$ ,

$$\min\{\varphi(x_{21}, y, \bar{\mu}), \varphi(x_{22}, z, \mu)\} \geq 0. \quad (8)$$

Putting  $y = \frac{1}{2}(x_{22} + x_{21})$  in [equation \(8\)](#), we have:

$$\varphi\left(x_{21}, \frac{x_{22} + x_{21}}{2}, \bar{\mu}\right) \geq 0.$$

By the strong convexity given in  $H2$ , we have:

$$f(x_{21}, x_{22}, \bar{\mu}) + f(x_{21}, x_{21}, \bar{\mu}) - \frac{1}{2}hd_2^\beta \geq 0. \tag{9}$$

The monotonicity of  $\varphi$  assumed in *H3* and  $x_{21} \in S(\lambda, \bar{\mu})$  imply that:

$$\varphi(x_{21}, x_{21}, \bar{\mu}) = 0$$

$$\varphi(x_{21}, x_{22}, \bar{\mu}) \leq -\varphi(x_{22}, x_{21}, \bar{\mu}).$$

Hence, combining with [equation \(9\)](#), we have:

$$\frac{1}{2}hd_2^\beta \leq -\varphi(x_{22}, x_{21}, \bar{\mu}). \tag{10}$$

Now, letting  $z = \frac{1}{2}(x_{22} + x_{21})$  in [equation \(8\)](#) derives:

$$\varphi\left(x_{22}, \frac{x_{22} + x_{21}}{2}, \mu\right) \geq 0.$$

Also by the strong convexity given in *H2*, we have:

$$\varphi(x_{22}, x_{22}, \mu) + \varphi(x_{22}, x_{21}, \mu) - \frac{1}{2}hd_2^\beta \geq 0.$$

As  $\varphi(x_{22}, x_{22}, \mu) = 0$ , this yields that:

$$\frac{1}{2}hd_2^\beta \leq \varphi(x_{22}, x_{21}, \mu).$$

Adding [equation \(10\)](#) and this inequality, and using assumption *H4*, we get:

$$\begin{aligned} hd_2^\beta &\leq f(x_{22}, x_{21}, \mu) - f(x_{22}, x_{21}, \bar{\mu}) \\ &\leq nd^\gamma(\bar{\mu}, \mu)d_2^\theta, \end{aligned}$$

i.e. we obtain [equation \(7\)](#).

*Step 3.* We are now ready to complete the proof. For all  $x_{11} \in S(\bar{\lambda}, \bar{\mu})$  and  $x_{22} \in S(\lambda, \mu)$ , we have:

$$d(x_{11}, x_{22}) \leq d_1 + d_2.$$

Hence, from [equations \(1\)](#) and [\(7\)](#), we get, with  $k_1 = \left(\frac{4m}{h}\right)^{\frac{1}{\beta}}$  and  $k_2 = \left(\frac{u}{h}\right)^{\frac{1}{\beta-\theta}}$ :

$$\rho\left(S(\bar{\lambda}, \bar{\mu}), S(\lambda, \mu)\right) \leq k_1 d^{\frac{\alpha}{\beta}}(\bar{\lambda}, \lambda) + k_2 d^{\frac{\gamma}{\beta-\theta}}(\bar{\mu}, \mu).$$

Letting  $\bar{\lambda} = \lambda$  and  $\bar{\mu} = \mu$  in this inequality, we see that the diameter of  $S(\bar{\lambda}, \bar{\mu})$  is 0 [for arbitrary  $(\bar{\lambda}, \bar{\mu})$ ], i.e. the solution map of (EP) is single-valued in  $\Lambda \times M$ . The proof is complete.

In the special case where  $K(\lambda) \equiv K$  ( $K$  is a nonempty set), we have the following result.

*Theorem 3.2.* For (EP) with  $K(\lambda) \equiv K$ , assume that assumptions  $H3$  and  $H4$  are satisfied and  $H2$  is replaced by the following condition:

$H'2$ : The map  $y \rightarrow \varphi(x, y, \mu)$  is  $h, \beta$ -strongly convex-like on  $K$ .

Then,  $S$  is Hölder calm on  $\Lambda \times M$ .

*Proof.* For any  $x_1 \in S(\bar{\mu})$ ,  $x_2 \in S(\mu)$  and  $y, z \in K$ , one has:

$$\min\{\varphi(x_1, y, \bar{\mu}), \varphi(x_2, z, \mu)\} \geq 0.$$

By the strong convex-likeness of  $f$  in  $K$ , there is  $\bar{z} \in K$  such that:

$$\varphi(x_1, \bar{z}, \bar{\mu}) \leq (1-t)\varphi(x_1, x_2, \bar{\mu}) + t\varphi(x_1, x_1, \bar{\mu}) - ht(1-t)d^\beta(x_1, x_2). \quad (11)$$

This and the monotonicity of  $\varphi$  imply that:

$$ht(1-t)d^\beta(x_1, x_2) \leq -(1-t)\varphi(x_2, x_1, \bar{\mu}) - \varphi(x_1, \bar{z}, \bar{\mu}).$$

As  $\varphi(x_1, \bar{z}, \bar{\mu}) \geq 0$ , this inequality leads to:

$$htd^\beta(x_1, x_2) \leq -\varphi(x_2, x_1, \bar{\mu}). \quad (12)$$

With the same arguments but with  $x_1$  replaced by  $x_2$  and  $\bar{\mu}$  by  $\mu$  in [equation \(11\)](#), we have:

$$h(1-t)d^\beta(x_1, x_2) \leq \varphi(x_2, x_1, \mu). \quad (13)$$

Adding [equations \(12\)](#) and [\(13\)](#), and using assumption  $H3$ , one gets:

$$hd^\beta(x_1, x_2) \leq \varphi(x_2, x_1, \mu) - \varphi(x_2, x_1, \bar{\mu})$$

$$\leq nd^\gamma(\bar{\mu}, \mu)d^\theta(x_1, x_2).$$

This means the required Hölder condition. From this, we obtain the uniqueness of the solution.

## 4. Applications

### 4.1 Mean-variance portfolio

In this subsection, we present an application of our results to a practical situation. Namely, we study Hölder properties of solution maps to parametric quadratic programming, a special case of the equilibrium problem, which has well-known applications in the view of practice. A good sample is the model of mean-variance portfolio based on investor's utility maximization. This problem can be formulated as parametric quadratic optimization problems. Consider a universe of  $n$  assets with the known data is  $c = (c_1, \dots, c_n)^T$  and  $Q = [\sigma_{ij}]$ , where  $c_i$  is the expected return for asset  $i$  and  $\sigma_{ij}$  is the covariance of returns for assets  $i$  and  $j$ . So,  $c$  is the vector of expected returns and  $Q$  is the  $n \times n$  variance-covariance matrix of asset returns. Herein,  $Q$  is a positive definite matrix which follows from the properties of variance-covariance matrices. Denote the vector of asset holdings by  $x = (x_1, \dots, x_n)^T$ . Then, the expected return of the portfolio  $x$  is  $c^T x$  and its variance is  $\sigma^2 = x^T Q x$ . A portfolio is said to be efficient if for some fixed level of expected return no

other portfolio gives smaller variance (Markowitz, 1956). In other words, an efficient portfolio can be considered as the one for which at some fixed level of variance no other portfolio gives larger expected return. The determination of the efficient portfolio frontier in this mean-variance model is equivalent to solving the following parametric problem because of (Farrar, 1965):

$$(MP) \quad \min f(x, \mu) := \frac{1}{2}x^T Qx - \mu c^T x \quad \text{s.t.} \quad Ax = b, x \in \mathbb{R}_+^n,$$

where  $\mu \geq 0$  is an investor's variance aversion parameter. The linear constraints  $Ax = b$  can represent budget constraints, bounds on asset holdings, etc. Nonnegativity constraints  $x \geq 0$  are short-sale constraints (non-negative asset holdings).

*Lemma 4.1.* If  $u(x)$  is  $h, \beta$ -strongly convex and  $v(x)$  is convex on a convex set  $A \subset \mathbb{R}^n$ , then  $g(x) = u(x) + v(x)$  is  $h, \beta$ -strongly convex on  $A$ .

*Proof.* For all  $x_1, x_2 \in A$  and  $t \in [0, 1]$ , we have:

$$\begin{aligned} g(tx_1 + (1-t)x_2) &= u(tx_1 + (1-t)x_2) + v(tx_1 + (1-t)x_2) \\ &\leq tu(x_1) + (1-t)u(x_2) - ht(1-t)d^\beta(x_1, x_2) + tv(x_1) \\ &\quad + (1-t)v(x_2) \\ &\leq t(u(x_1) + v(x_1)) + (1-t)(u(x_2) + v(x_2)) \\ &\quad - ht(1-t)d^\beta(x_1, x_2) \\ &\leq tg(x_1) + (1-t)g(x_2) - ht(1-t)d^\beta(x_1, x_2). \end{aligned}$$

Hence,  $g$  is a strongly convex function.

*Lemma 4.2.* The function  $g(x) = x^T Qx$  is strongly convex with respect to the Euclidean norm in  $\mathbb{R}^n$ , where  $Q$  is a positive definite matrix.

*Proof.* As  $Q$  is a positive definite matrix, we can consider the norm defined by  $\|x\| = \sqrt{x^T Qx}$  on  $\mathbb{R}^n$ . We first prove that the function  $g(x) = x^T Qx$  is strongly convex with respect to this norm. Indeed, for all  $x_1, x_2 \in \mathbb{R}^n$  and  $t \in [0, 1]$ , one has:

$$\begin{aligned} g(tx_1 + (1-t)x_2) &= (tx_1 + (1-t)x_2)^T Q(tx_1 + (1-t)x_2) \\ &= t^2 x_1^T Qx_1 + t(1-t)x_1^T Qx_2 + t(1-t)x_2^T Qx_1 \\ &\quad + (1-t)^2 x_2^T Qx_2 \\ &= t^2 x_1^T Qx_1 + (1-t)^2 x_2^T Qx_2 + 2t(1-t)x_1^T Qx_2 \end{aligned}$$

$$\begin{aligned}
 &= tx_1^T Qx_1 + (1-t)x_2^T Qx_2 \\
 &\quad -t(1-t)\left(x_1^T Qx_1 + x_2^T Qx_2 - 2x_1^T Qx_2\right) \\
 &= tx_1^T Qx_1 + (1-t)x_2^T Qx_2 - t(1-t)\|x_1 - x_2\|^2 \\
 &= tg(x_1) + (1-t)g(x_2) - t(1-t)\|x_1 - x_2\|^2.
 \end{aligned}$$

We now show that  $g(x) = x^T Qx$  is strongly convex with respect to the Euclidean norm  $\|\cdot\|_E$ . As any two norms on a finite dimensional normed space are equivalent, there are  $a, b > 0$  such that  $a\|\cdot\|_E \leq \|\cdot\| \leq b\|\cdot\|_E$ . Therefore, for all  $x_1, x_2 \in \mathbb{R}^n$  and  $t \in [0, 1]$ , we get:

$$g(tx_1 + (1-t)x_2) \leq tg(x_1) + (1-t)g(x_2) - a^2t(1-t)\|x_1 - x_2\|_E^2.$$

The proof is complete.

If we set  $\varphi(x, y, \mu) := f(y, \mu) - f(x, \mu)$ , then (MP) becomes the parametric equilibrium problem. For  $\mu \geq 0$ , we denote the solution set of (MP) by  $S_1(\mu)$ .

*Lemma 4.3.* If the function  $y \mapsto g(y)$  is strongly convex, then the function  $y \mapsto \varphi(x, y) := f(y) - f(x)$  is also strongly convex.

*Proof.* We omit the proof as it is trivial.

We have the result for the Hölder calmness of the solution map  $S_1(\mu)$  as follows:

*Corollary 4.1.* If  $Q$  is a positive definite matrix then the solution map  $S_1$  is Hölder calm on  $(0, +\infty)$ .

*Proof.* We prove this corollary by checking all assumptions of Theorem 3.1. We see that assumptions  $H1, H3$  and  $H4$  are obviously fulfilled. The strong convexity and Hölder calmness in  $H2$  is derived from Lemmas 4.1, 4.2 and 4.3.

#### 4.2 Nash equilibrium problem

In this subsection, we discuss an application of the main results to a Nash equilibrium problem (Debreu, 1952; Facchinei and Kanzow, 2007). Let us state the model of the problem as follows. Let  $I \subset \mathbb{N}$  be an index set, and  $\Lambda, M, X_i$  ( $i \in I$ ) be linear metric vector spaces.

We use the notations  $X = \prod_{i \in I} X_i$  and  $X_{-i} = \prod_{j \in I, j \neq i} X_j$ . For each  $x \in X$ , we denote its  $i$ th

coordinate and its projection on  $X_i$  by  $x_i$  and  $x_{-i}$ , respectively. We also denote an element of  $X$  by  $(x_i, x_{-i})$ . For  $i \in I$ , let  $f_i$  be extended real-valued functions defined on  $X \times M$  and  $K_i$  be set-valued mappings from  $\Lambda$  into  $X_i$ . For any fixed  $(\lambda, \mu) \in \Lambda \times M$ , an abstract economy, introduced by Debreu (1952), is the set of data:

$$\Gamma(\lambda, \mu) = \{X_i, K_i(\lambda), f_i(\cdot, \mu) : i \in I\}.$$

A profile of strategies  $x^* \in X$  is said to be a social Nash equilibrium of  $\Gamma(\lambda, \mu)$  if for any  $i \in I, x^* \in K(\lambda)$  and:

$$f_i(x^*, \mu) = \max_{y_i \in K(\lambda)} f_i\left(\left(y_i, x_{-i}^*\right), \mu\right). \quad (14)$$

For each  $(\lambda, \mu) \in \Lambda \times M$ , the set of all social Nash equilibria of the parametric abstract economy  $\Gamma(\lambda, \mu)$  is denoted by  $\mathcal{N}(\lambda, \mu)$ . The set-valued mapping  $\mathcal{N} : \Lambda \times M \rightrightarrows X$  is called a social Nash equilibria set-valued mapping. The following lemma gives us the equivalence between the social Nash equilibrium problem and (EP).

*Lemma 4.4.* For given  $(\bar{\lambda}, \bar{\mu}) \in \Lambda \times M$ ,  $x^*$  is a Nash equilibrium of abstract economy  $\Gamma(\bar{\lambda}, \bar{\mu})$  if and only if  $x^* \in K(\bar{\lambda})$  and:

$$\varphi(x^*, y, \bar{\mu}) \geq 0, \quad \forall y \in K(\bar{\lambda}), \tag{15}$$

i.e.  $x^*$  is a solution to (EP), where:

$$K(\bar{\lambda}) := \prod_{i \in I} K_i(\bar{\lambda}) \text{ and } \varphi(x, y, \bar{\mu}) = \sum_{i \in I} [f_i((x_i, x_{-i}), \bar{\mu}) - f_i((y_i, x_{-i}), \bar{\mu})].$$

*Proof.* If [equation \(14\)](#) holds for  $\bar{\lambda} \in \Lambda$ , then it is obvious that  $x^*$  is a solution to (EP). Conversely, let  $x^* \in X$  be such that  $x^* \in K(\bar{\lambda})$  and [equation \(15\)](#) holds. For each  $i \in I$ , we choose  $\hat{y} \in X$  in such a way satisfying  $\hat{y}_i = y_i \in K_i(\bar{\lambda})$  and  $\hat{y}_j = x_j^*$  for all  $j \in I \setminus \{i\}$ . Then,  $\hat{y} \in K(\bar{\lambda})$  and:

$$\begin{aligned} \varphi(x^*, \hat{y}, \bar{\mu}) &= \sum_{i \in I} [f_i((x_i^*, x_{-i}^*), \bar{\mu}) - f_i((\hat{y}_i, x_{-i}^*), \bar{\mu})] \\ &= \sum_{\substack{j \neq i \\ j \in I}} [f_j((x_j^*, x_{-j}^*), \bar{\mu}) - f_j((\hat{y}_j, x_{-j}^*), \bar{\mu})] + f_i((x_i^*, x_{-i}^*), \bar{\mu}) - f_i((\hat{y}_i, x_{-i}^*), \bar{\mu}) \\ &= f_i((x_i^*, x_{-i}^*), \bar{\mu}) - f_i((\hat{y}_i, x_{-i}^*), \bar{\mu}). \end{aligned}$$

Combining this with [equation \(15\)](#), we derive  $f_i((x_i^*, x_{-i}^*), \bar{\mu}) \geq f_i((y_i, x_{-i}^*), \bar{\mu})$  for all  $y_i \in K_i(\bar{\lambda})$  and  $i \in I$ . Consequently, we conclude that for each  $i \in I$ ,  $x^* \in K(\bar{\lambda})$  and  $f_i(x^*, \bar{\mu}) = \max_{y_i \in K_i(\bar{\lambda})} f_i((y_i, x_{-i}^*), \bar{\mu})$ , i.e.  $x^*$  is a social Nash equilibrium of abstract economy  $\Gamma(\bar{\lambda}, \bar{\mu})$ .

Lemma 4.4 allows us to apply Theorem 3.1 to the mapping  $\mathcal{N}$ , obtaining the following result.

*Theorem 4.1.* Assume that the set-valued map  $K : \Lambda \rightrightarrows X$ , and the function  $\varphi : X \times X \times \Lambda \rightarrow \mathbb{R} \cup \{\infty\}$  defined by:

$$K(\lambda) := \prod_{i \in I} K_i(\lambda),$$

and:

$$\varphi(x, y, \lambda) := \sum_{i \in I} [f_i((x_i, x_{-i}), \mu) - f_i((y_i, x_{-i}), \mu)],$$

respectively, satisfy conditions (i)-(iv) of Theorem 3.1 on  $\Lambda \times M$ . Then, the set-valued mapping  $\mathcal{N}$  is Hölder calm on  $\Lambda \times M$ .

## 5. Conclusions

In this paper, we are successful in establishing the Hölder calmness of solution maps to equilibrium problems where both objective functions and constraints are perturbed. Our results are new. These main results are applied to some special cases including mean-variance portfolio and Nash equilibrium problem. Tools used in this paper can be used for investigating more generalized settings.

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